

Consider a δ -function as a narrow "top-hat" function; width 2ϵ , height $1/2\epsilon$, with $\epsilon \rightarrow 0$.

$$\begin{aligned} \text{Then } & \int \delta(x-x_0) \cdot g(x) dx \\ & \text{across } x_0 \\ & = \int_{x_0-\epsilon}^{x_0+\epsilon} \frac{1}{2\epsilon} \cdot g(x) dx. \end{aligned}$$

This is the average of $g(x)$ over $x_0 - \epsilon$ to $x_0 + \epsilon$.

As $\epsilon \rightarrow 0$ this is $g(x_0)$.

