Borwein integral(s)

In 2001 Borwein & Borwein noted that the pattern

$$\int_{-\infty}^{\infty} \operatorname{sinc}(t)\,\mathrm{d}t = \pi, \tag{B.1}$$

$$\int_{-\infty}^{\infty} \operatorname{sinc}(t)\operatorname{sinc}\left(\frac{t}{3}\right)\,\mathrm{d}t = \pi, \tag{B.2}$$

$$\int_{-\infty}^{\infty} \operatorname{sinc}(t)\operatorname{sinc}\left(\frac{t}{3}\right)\operatorname{sinc}\left(\frac{t}{5}\right)\,\mathrm{d}t = \pi, \tag{B.3}$$

continues up to

$$\int_{-\infty}^{\infty} \operatorname{sinc}(t)\operatorname{sinc}\left(\frac{t}{3}\right)\,\ldots\,\operatorname{sinc}\left(\frac{t}{13}\right)\,\mathrm{d}t = \pi, \tag{B.4}$$

but then

$$\int_{-\infty}^{\infty} \operatorname{sinc}(t)\operatorname{sinc}\left(\frac{t}{3}\right)\,\ldots\,\operatorname{sinc}\left(\frac{t}{15}\right)\,\mathrm{d}t \approx \pi - 4.6\times 10^{-11}. \tag{B.5}$$

These results can be understood using Fourier transforms and convolution.

Fourier transforms between $f(t)$ and $F(\omega)$ are

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega, \quad \text{and} \quad F(\omega) = \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t. \tag{B.6}$$

The Fourier transform of the product of two functions $f(t)g(t)$ is

$$H(\omega) = \int_{-\infty}^{\infty} f(t)g(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t, \tag{B.7}$$

and it can be shown that

$$H(\omega) = \frac{1}{2\pi} F(\omega) * G(\omega), \tag{B.8}$$

i.e. $1/2\pi$ times the convolution of $F(\omega)$ and $G(\omega)$.
Define the $\operatorname{rect}(t)$ function – a 'top-hat' function, which has unit area – as

$$\operatorname{rect}(t) = \begin{cases} 0 & \text{if } |t| > 1/2; \\ 1/2 & \text{if } |t| = 1/2; \\ 1 & \text{if } |t| < 1/2. \end{cases} \tag{B.9}$$

Consider

$$F_n(\omega) = 2\pi\left(\frac{n}{2}\right)\operatorname{rect}\left(\frac{n\omega}{2}\right), \tag{B.10}$$

which is 2π times a top-hat function of height $n/2$ and width $2/n$, i.e. an area of unity. Then its Fourier transform is

$$f_n(t) = \operatorname{sinc}\left(\frac{t}{n}\right). \tag{B.11}$$

Now consider

$$H(\omega) = \int_{-\infty}^{\infty} \operatorname{sinc}(t)\operatorname{sinc}\left(\frac{t}{3}\right)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t, \tag{B.12}$$

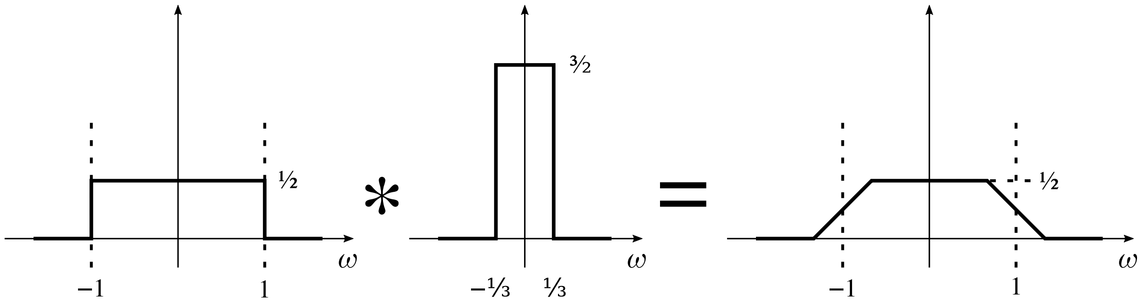
which is the FT of $\operatorname{sinc}(t)\times\operatorname{sinc}(t/3)$. Then the LHS of B.2 above is

$$\int_{-\infty}^{\infty} \operatorname{sinc}(t)\operatorname{sinc}\left(\frac{t}{3}\right)\,\mathrm{d}t = H(0). \tag{B.13}$$

From B.7, put $f(t) = \text{sinc}(t) = f_1(t)$ and $g(t) = \text{sinc}(t/3) = f_3(t)$, so using B.11, then B.8 gives

$$H(0) = \frac{1}{2\pi} F_1(\omega) * F_3(\omega)\Big|_{\omega=0} = 2\pi \left[\underbrace{\frac{1}{2}\,\text{rect}\left(\frac{\omega}{2}\right)}_{\text{wide top-hat}} * \underbrace{\frac{3}{2}\,\text{rect}\left(\frac{3\omega}{2}\right)}_{\text{narrow top-hat}} \right]_{\omega=0} . \qquad \text{B.14}$$

The '[...]', is the convolution of a wide top-hat with a narrow top-hat. Note the height of the wide top-hat is ½, and each top-hat has unit area.



At $\omega = 0$ this convolution is ½ (because the width of the narrow top-hat is less than the width of the wide top-hat), so $H(0) = \pi$. Thus B.2 *is*

$$\int_{-\infty}^{\infty} \text{sinc}(t)\,\text{sinc}\left(\frac{t}{3}\right)\,\mathrm{d}t = \pi. \qquad \text{B.15}$$

Note that the flat region of this convolution, of height ½, is now shorter that the width of the wide top-hat (by the width of the narrow top-hat).

For the next step, B.3, multiplying with $\text{sinc}(t/5)$ means another convolution with an even narrower top-hat function. At $\omega = 0$ this still has a value of ½, because $\frac{1}{3} + \frac{1}{5} < 1$, as the result from all the convolutions is still flat near $\omega = 0$.

This continues up to B.4, including the $\text{sinc}(t/13)$ term, as $\frac{1}{3} + \frac{1}{5} \cdots \frac{1}{13} < 1$. However, for B.5, including up to the $\text{sinc}(t/15)$ term, the value of all the convolutions at $\omega = 0$ is a bit less than ½, because $\frac{1}{3} + \frac{1}{5} \cdots \frac{1}{15} > 1$, so the result from all the convolutions is no longer flat at $\omega = 0$.